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Domination dot-critical graphs

Tamara Burton[a,1], David P. Sumner[b]

[a]*Department of Mathematics and Statistics, Rochester Institute of Technology, Rochester, NY 14623, USA*
[b]*Mathematics Department, University of South Carolina, Columbia, SC 29208, USA*

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Abstract

A graph G is dot-critical if contracting any edge decreases the domination number. It is totally dot-critical if identifying any two vertices decreases the domination number. We show that the totally dot-critical graphs essentially include the much-studied domination vertex-critical and edge-critical graphs as special cases. We investigate these properties, and provide a characterization of dot-critical and totally dot-critical graphs with domination number 2. We also consider the question of when a dot-critical graph contains a critical vertex.

Keywords: Tree; Domination; Critical; Corona

1. Introduction and preliminaries

A set of vertices S in a graph G is a *dominating* set if every vertex of $G-S$ is adjacent to some vertex of S. If S has the smallest possible cardinality of any dominating set of G, then S is called a *minimum dominating set*—abbreviated MDS. The cardinality of any MDS for G is called the *domination number* of G and is denoted by $\gamma(G)$. More generally, we say that a set of vertices A *dominates* the set B if every vertex of $B-A$ is adjacent to some vertex in A. Two graphs A and B are *disjoint* if they have no vertices in common and no vertex of A is adjacent to any vertex of B. We denote the neighborhood of a vertex v by $N(v)$ and its closed neighborhood by $N[v]$ (so we have $N[v]=N(v)\cup\{v\}$). We denote the degree of a vertex v by $\deg(v)$. We indicate that v is adjacent to u by writing $v\perp u$. We denote the edge with endpoints v and u by vu. We denote the complement of the graph G by $\bar{G}$. We write $d(v,u)$ for the distance between the vertices v and u.

When no confusion is possible, we do not distinguish between a set A of vertices and the subgraph that it induces. For instance, if S is a set of vertices of the graph G and A is a subgraph of G, then we may write $S\cap A$ instead of $S\cap V(A)$. For terminology not defined in this section, see West [6].

If a property of graphs is worth studying, then it is almost certainly worthwhile to investigate those graphs that are extreme with respect to that property. But there may be many ways in which a graph can be extreme. In particular,

E-mail address: sumner@math.sc.edu (D. Sumner).
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for the domination number, there are a variety of extremal concepts that have been investigated. The two most-studied are the edge-critical graphs introduced by Sumner and Blitch [4] and the vertex-critical graphs introduced by Brigham et al. [1].

A graph G is *edge-critical* with respect to the domination number if for every two non-adjacent vertices v and u, $\gamma(G+vu)<\gamma(G)$. A vertex v of G is *critical* if $\gamma(G-v)<\gamma(G)$. A graph G is *vertex-critical* if every vertex of G is critical. We denote the set of critical vertices of G by G'. A vertex v is *stable* if $\gamma(G-v)=\gamma(G)$.

In this paper we introduce a new critical condition for the domination number. A graph is *domination dot-critical* (hereafter, just *dot-critical*) if identifying any two adjacent vertices (i.e., contracting the edge comprising those vertices) results in a graph with smaller domination number. If identifying *any* two vertices of G causes the domination number to decrease, then we say that G is *totally dot-critical*. In the next section of this paper we formally define these terms and show that the totally dot-critical graphs contain the vertex-critical and essentially (in a sense made precise in Section 2) the edge-critical graphs as special cases.

When we say that G is k-edge-critical, k-vertex-critical, k-dot-critical, or totally-k-dot-critical, we mean that it has the indicated property and that $\gamma(G)=k$.

When we first began studying dot-critical graphs, we generated numerous examples at random by use of a computer. We were struck by the fact that all the generated examples were dominated by the set of their critical vertices and in many instances that the set of critical vertices was a minimum dominating set. This motivated us to define a graph to be *critically dominated* if its set of critical vertices forms a dominating set.

For a brief period it seemed that perhaps all dot-critical graphs were critically dominated. But that is not the case in general. In fact we have examples of dot-critical graphs with no critical vertices at all. Still, the question looms as to under what conditions a dot-critical graph is critically dominated as well as the more basic question of when it contains a critical vertex at all. In Section 3 we discuss the question of when a dot-critical graph contains a critical vertex.

A Java applet version of the program we used to study these properties is available online at http://www.math.sc.edu/∼sumner/graphlet/index.html. A manual explaining the various options of this applet is available from the second author. Note that the applet works best with Internet Explorer version 5.0 or better and of course with Java enabled. Some of the results in this paper appear also in the dissertation of the first author [2] (directed by the second author).

2. Dot-critical graphs

For a pair of vertices v, u of G, we denote by $G.vu$ the graph obtained by identifying v and u. We let (vu) denote the identified vertex. So $G.vu$ may be viewed as the graph obtained from G by deleting the vertices v and u and appending a new vertex, denoted by (vu), that is adjacent to all the vertices of $G-v-u$ that were originally adjacent to either of v or u. In the case that v is adjacent to u, $G.vu$ is the graph obtained by contracting vu. A graph G is *dot-critical* if $\gamma(G.vu)<\gamma(G)$ for any two adjacent vertices v and u. Since it is apparent that identifying vertices cannot cause the domination number to decrease by more than one, G is dot-critical if and only if $\gamma(G.vu)=\gamma(G)-1$ for any two adjacent vertices v and u. Similarly, a graph is *totally dot-critical* if and only if for *any* two vertices v and u, $\gamma(G.vu)=\gamma(G)-1$. Clearly, every totally dot-critical graph is dot-critical. The following lemmas are fundamental.

Lemma 2.1. *Let $a, b \in V(G)$ for a graph G. Then $\gamma(G.ab)<\gamma(G)$ if and only if either there exists an MDS S of G such that $a, b \in S$ or at least one of a or b is critical in G.*

Proof. $\Rightarrow$ Let $a, b \in V(G)$ such that $\gamma(G.ab)<\gamma(G)$. Let S be an MDS of $G.ab$. If $(ab)\in S$, then $S^*=[S-(ab)]\cup\{a,b\}$ is an MDS of G containing a and b. If $(ab)\notin S$, then there exists a $t \in S$ such that $t \perp (ab)$. If $t \in N(a)\cap N(b)$, then S dominates G, which contradicts that $\gamma(G.ab)<\gamma(G)$. Thus t is adjacent to exactly one of a or b in G, say $t \perp a$. Then S dominates $G-b$ which implies that $b \in G'$.

$\Leftarrow$ If a, b belong to a common MDS S of G, then $S^*=(S-\{a,b\})\cup(ab)$ is a dominating set of $G.ab$ of cardinality $\gamma(G)-1$. On the other hand, if $a \in G'$, then any MDS of $G-a$ is a dominating set of $G.ab$ of cardinality $\gamma(G)-1$. Hence $\gamma(G.ab)<\gamma(G)$. □

The next lemma is an immediate consequence of Lemma 2.1.

Lemma 2.2. *If G is any graph with $\gamma(G)=k\geqslant 2$, then G is dot-critical (resp. totally dot-critical) if and only if every two adjacent non-critical vertices (resp. any two non-critical vertices) belong to a common MDS.*

A vertex in a graph G is *useable* if it belongs to some MDS of G. If every vertex of G is useable, then we say that G is *vertex-useable*.

Lemma 2.3. *Let G be any graph, and $v \in G'$. Then all of $N[v]$ is useable.*

Proof. Let S be an MDS of $G-v$ and $u \in N(v)$. Then $S \cup \{v\}$ and $S \cup \{u\}$ are both MDS's of G. □

Theorem 2.4. *For every graph G,*

1. *If G is dot-critical, then G is vertex-useable.*
2. *If G is critically dominated, then G is vertex-useable.*

Proof. (1) Let $v \in V(G)$ and u any neighbor of v. Then if v or u is critical, then v is useable by Lemma 2.3. Otherwise there is an MDS that contains both v and u and again v is useable.

(2) This is immediate from Lemma 2.3. □

As a consequence of the next result, we will generally limit our discussion to graphs that are connected.

Lemma 2.5. *G is dot-critical (resp. totally dot-critical) if and only if each of its components is dot-critical (resp. totally dot-critical).*

Proof. This lemma is clear for dot-critical graphs, and it is also apparent that every component of a totally dot-critical graph is also totally dot-critical. So suppose that each component of G is totally dot-critical. We argue that G itself is totally dot-critical. Clearly, identifying any two vertices in the same component of G decreases $\gamma(G)$. So suppose that x, y belong to separate components of G. Then by Theorem 2.4 we may select an MDS S_1 of the component containing x so that $x \in S_1$ and an MDS S_2 of the component containing y so that $y \in S_2$. Let S be an MDS for the remaining components, if any. Then $S^* = S_1 \cup S_2 \cup S$ is an MDS that contains both x and y. □

A graph G is *point-distinguishing* if every two distinct vertices have distinct closed neighborhoods. It is a consequence of the following simple lemma that every dot-critical graph is point-distinguishing.

Lemma 2.6. *If $v, u \in V(G)$ for a graph G such that $N[v]=N[u]$, then $\gamma(G)=\gamma(G.vu)$.*

Proof. Suppose that $\gamma(G.vu)<\gamma(G)$. Then by Lemma 2.2 either one of $v, u \in G'$ or there is an MDS S of G containing v and u. Suppose that $v \in G'$, and let S_v be an MDS of $S-v$ of cardinality $\gamma(G)-1$. Then any vertex of S_v that is adjacent to u in G is also adjacent to v in G. This implies that S_v dominates G. Thus, neither v nor u is critical in G. If S is a dominating set of G with v, u both in S, then $S-u$ is also a dominating set of G. Hence, no MDS of G may contain both v and u, and so $\gamma(G)=\gamma(G.vu)$. □

It is apparent from Lemma 2.2 that every vertex-critical graph is also totally dot-critical. Although not every edge-critical graph is totally dot-critical, the next theorem shows that the point-distinguishing edge-critical graphs are totally dot-critical.

Theorem 2.7. *Every point-distinguishing, edge-critical graph is totally dot-critical.*

Proof. Let G be a point-distinguishing, k-edge-critical graph and $x, y \in V(G)$. Let H be the graph with $V(H)=V(G)$ and having its edge set defined by $E(H)=E(G)\cup\{xa : a \in N(y)\}\cup\{yb : b \in N(x)\}\cup\{xy\}$. Since G is point-distinguishing, at least one new edge was added to make H (perhaps the edge xy if x and y were non-adjacent). G is

k-edge-critical, and so $\gamma(H) \leqslant k-1$. Since $N_H[x]=N_H[y]$, Lemma 2.6 implies that $\gamma(H)=\gamma(H.xy)$. Also $H.xy \cong G.xy$ and so $k-1 \geqslant \gamma(H)=\gamma(H.xy)=\gamma(G.xy) \geqslant k-1$. Hence $\gamma(G.xy)=k-1$ and so G is k-totally dot-critical. □

Since every edge-critical graph reduces to a point-distinguishing one by identifying vertices with the same closed neighborhood (see [5]), it follows that for most purposes the edge-critical graphs may be viewed as a subclass of the totally dot-critical ones. Although the totally dot-critical graphs include all the vertex-critical and point-distinguishing edge-critical graphs as special cases, there are a multitude of totally dot-critical graphs that do not belong to either of these classes.

A vertex v is *selfish* in the MDS S if v is needed only to dominate itself. More precisely, v is selfish in S means that for any vertex $x \notin S$ that is dominated by v, x is also dominated by some vertex of S other than v. In the terminology of [3] a vertex is selfish if and only if $pn(v, S)=\{v\}$ (i.e., v is its only private neighbor). The following simple lemma often simplifies the presentation of our arguments.

Lemma 2.8. *For any graph G, $v \in G'$ if and only if there is some MDS S of G in which v is selfish.*

Let A and B be disjoint graphs and a and b vertices of A and B, respectively. Brigham et al. [1] define the *coalescence* G of A and B (with respect to a and b) to be the graph obtained by identifying a and b. They show that G is vertex-critical if and only if each of A and B is vertex-critical. We extend this result in the following theorem, which provides a mechanism for producing larger dot-critical graphs from smaller ones.

Theorem 2.9. *Let A and B be disjoint graphs with $\gamma(A)=n$, $\gamma(B)=m$, and a, b critical vertices of A and B, respectively. Let G be the coalescence obtained by identifying a and b. Then*

1. $\gamma(G)=n+m-1$.
2. $G'=(A' \cup B' - \{a, b\}) \cup \{(ab)\}$.
3. *G is dot-critical if and only if each of A and B is dot-critical.*
4. *G is vertex-critical if and only if each of A and B is vertex-critical.*
5. *G is critically dominated if and only if each of A and B is critically dominated.*
6. *G is vertex-useable if and only if each of A and B is vertex-useable.*

Proof. We first establish parts (1) and (2). Let S_a and S_b be MDSs of $A-a$ and $B-b$, respectively. Then $S=S_a \cup S_b \cup \{(ab)\}$ is a dominating set of G of cardinality $n+m-1$. Let H denote the disjoint union of A and B. Since identifying vertices cannot decrease the domination number by more than one, $\gamma(G)=\gamma(H.ab) \geqslant \gamma(H)-1 \geqslant n+m-1$. Thus $\gamma(G)=n+m-1$. Next we will show that $G'=(A' \cup B' - \{a, b\}) \cup \{(ab)\}$. $G-(ab)$ is isomorphic to the disjoint union of $A-a$ and $B-b$. So $\gamma(G-(ab))=\gamma(A-a)+\gamma(B-b)=n+m-2$. Hence $(ab) \in G'$. Suppose that $v \in A'-a$, and let S_v be an MDS of $A-v$. Then S_v dominates the vertex (ab) in $G-v$. Let S_b be an MDS of $B-b$. Then $S=S_v \cup S_b$ is a dominating set of $G-v$ of cardinality $n+m-2$. Hence $v \in G'$. So $A'-\{a\} \subseteq G'$ and similarly $B'-\{b\} \subseteq G'$ and so $A' \cup B' - \{a, b\} \subseteq G'$.

The following observation is straightforward to verify and we omit the details.

Observation. If S is an MDS for G, then either $S \cap A$ or $(S \cap A) \cup \{a\}$ is an MDS for A (similarly for B and b).

Let $v \in G'-(ab)$. Without loss of generality, suppose that $v \in A-a$. Let S be an MDS of G in which v is selfish. Then by the observation, one of $S \cap A$ or $(S \cap A) \cup \{a\}$ is an MDS for A, and it is simple to see that v is still selfish in either case. Hence, $v \in A'$.

(3) Now suppose that each of A and B is dot-critical. Let x and y be adjacent non-critical vertices of G. We show that there is an MDS of G that contains x and y. Since x and y are adjacent and non-critical, they both belong to one of A or B. With no loss of generality, suppose that x and y belong to A. Then by (2) we know that x and y are not critical in A. Hence since A is dot-critical, we may find an MDS S_A of A that contains both x and y. If $a \in S_A$, then replace S_A by $(A_A - a) \cup \{(ab)\}$. But now letting S_B denote any dominating set for $B-b$, we get that $S_A \cup S_B$ is an MDS of G that contains both x and y. Thus, G is dot-critical whenever each of A and B is dot-critical.

Now suppose that G is dot-critical, and let x and y denote adjacent non-critical vertices of A. Then by (2) above, x and y are non-critical vertices in G, and so by Lemma 2.2 there exists an MDS S of G that contains both x and y. But then by the observation, one of $S \cap A$ or $(S \cap A) \cup \{a\}$ is an MDS for A that contains both x and y. Thus, A is dot-critical. Similarly, B is dot-critical.

The remaining parts of the theorem are direct consequences of the observation and parts (1) and (2). □

Note that part (4) of Theorem 2.9 is the result of Brigham et al. [1]. Unfortunately, totally dot-critical graphs and edge-critical graphs are not generally preserved by this operation.

3. 2-Dot-critical graphs

The trivial graph on one vertex is clearly the only 1-dot-critical graph and henceforth we assume that all our graphs have domination number at least two.

Sumner and Blitch [4] characterized the 2-edge-critical graphs, and Brigham et al. [1] characterized the 2-vertex-critical graphs. In both cases the characterization can be expressed conveniently in terms of the structure of $\bar{G}$.

Theorem 3.1. *Let G be a graph. Then*

1. *G is 2-vertex-critical if and only if every component of $\bar{G}$ is K_2.*
2. *G is 2-edge-critical if and only if every component of $\bar{G}$ is a star.*

The structure of 2-dot-critical graphs is a bit more complex than that of the edge-critical and vertex-critical graphs. However, our characterization of 2-dot-critical graphs is also in terms of the structure of the complement.

Theorem 3.2 characterizes the critical vertices of a graph G with $\gamma(G) = 2$.

Theorem 3.2. *Let G be a graph with $\gamma(G) = 2$. Then the critical vertices of G are precisely those, which are adjacent to an end vertex in $\bar{G}$.*

Proof. Let $x \in V(G)$ such that x is an end vertex in $\bar{G}$. If $x \perp y$ in $\bar{G}$, then x dominates $G - y$, and hence $y \in G'$. On the other hand, suppose that $y \in G'$ and that x dominates $G - y$. Then in $\bar{G}$, x is an end-vertex adjacent to y. □

We will make frequent use of the following lemma.

Lemma 3.3. *Let a, b and v be vertices of the 2-dot-critical graph G such that in $\bar{G}$, $v \perp a$, $v \perp b$, and a is not adjacent to b. Then*

1. *One of a, b is adjacent to an end vertex in $\bar{G}$.*
2. *v is adjacent to an end vertex of $\bar{G}$.*

Proof. (1) In G, v is not adjacent to either of a or b, so $\{a, b\}$ is not a dominating set of G. So by Lemma 2.2, at least one of a or b belongs to G' and hence, by Theorem 3.2, is adjacent to an end vertex in $\bar{G}$.

(2) From part (1) we may assume without loss of generality that b is adjacent to an end vertex z in $\bar{G}$. Thus in $\bar{G}$, b is adjacent to both of z and v, and z is not adjacent to v. Hence by (1), at least one of v or z is adjacent to an end vertex in $\bar{G}$. Since z *is* an end vertex in $\bar{G}$, it follows that v is adjacent to an end vertex in $\bar{G}$. □

A graph G is said to be *spiked* if $G = H \circ K_1$, the corona of a connected graph H with a single vertex. So G is spiked if it is non-trivial, connected, and every vertex of G is either an end vertex or is adjacent to exactly one end vertex.

Note that K_2 is both a complete graph and spiked. Also note that $\bar{K}_2$ is (vacuously) the only 2-dot-critical graph on $n \leqslant 3$ vertices.

Theorem 3.4. *Let G be a graph on $n \geqslant 4$ vertices. Then G is 2-dot-critical if and only if $\bar{G}$ is not complete, but every component of $\bar{G}$ is spiked or a complete graph K_m, $m \geqslant 2$.*

Proof. $\Leftarrow$ Suppose that every component of $\bar{G}$ is one of the two types of graphs above. First, we argue that $\gamma(G)=2$. If $\gamma(G)=1$, then $\bar{G}$ contains an isolated vertex which is not either of the two allowable types of components. Thus, $\gamma(G)\geqslant 2$. If $\bar{G}$ has at least two components, then any two vertices from different components in $\bar{G}$ form a dominating set of G. If $\bar{G}$ has exactly one component, then $\bar{G}$ is spiked, and thus $\bar{G}$ contains at least two end vertices, which form a dominating set of G. Hence $\gamma(G)=2$. Now we argue that G is dot-critical. Let $vu \in E(G)$. Then by Lemma 2.2, it is enough to show that either $\{v, u\}$ is a MDS of G or one of v or u is critical. So suppose $\{v, u\}$ does not dominate G. Then there exists a vertex $x \in V(G)$ such that x is not adjacent to either of v or u. Thus in $\bar{G}$, $x \perp v$ and $x \perp u$. So v and u belong to the same component, C, in $\bar{G}$. Also v is not adjacent to u in C, and so C is spiked. Since $x \perp v$ and $x \perp u$ in $\bar{G}$, not both of v and u are end vertices in $\bar{G}$. Suppose $\deg(v)\geqslant 2$ in $\bar{G}$. Then v is adjacent to an end vertex in $\bar{G}$ and so from Theorem 3.2 it follows that $v \in G'$.

$\Rightarrow$ Now suppose that G is a 2-dot-critical graph. Let C be a component of $\bar{G}$ and assume C is not complete. Let v be any vertex in C. Suppose v is not an end vertex of $\bar{G}$. We must show that v is adjacent to exactly one end vertex of $\bar{G}$. By Lemma 3.3, no vertex of $\bar{G}$ can be adjacent to two end vertices, and thus it is enough to show that v is adjacent to at least one end vertex. Since v is not an end vertex in $\bar{G}$, v has at least two neighbors in $\bar{G}$.

Case 1: For some $a, b \in N(v)$, a is not adjacent to b.

By Lemma 3.3, v is adjacent to an end vertex in $\bar{G}$.

Case 2: The neighborhood of v is complete in $\bar{G}$.

Since C is not complete, there exists a vertex $t \in V(\bar{G})$ such that $d(t, v)=2$. Let r be a common neighbor of v and t. Then by Lemma 3.3, r is adjacent to an end vertex, say s. Since $r \perp s$ and $r \perp v$ but s is not adjacent to v, then again by Lemma 3.3 either s or v is adjacent to an end vertex in $\bar{G}$. Hence, since s is an end vertex, it must be v that is adjacent to an end vertex in $\bar{G}$, but this is impossible. $\square$

The next result characterizes the totally 2-dot-critical graphs on $n\geqslant 2$ vertices.

Theorem 3.5. *G is a totally 2-dot-critical graph on $n\geqslant 2$ vertices if and only if every component of $\bar{G}$ is spiked.*

Proof. $\Leftarrow$ By the proof of Theorem 3.4, we know these graphs are 2-dot-critical. So let a, b be two non-adjacent vertices of G. If $\{a, b\}$ induces a K_2 in $\bar{G}$, then $\{a, b\}$ dominates G, and otherwise $a \perp b$ in $\bar{G}$ and a and b are not both end vertices in $\bar{G}$. Thus one of a, b, say a, is adjacent to an end vertex in $\bar{G}$ and so by Theorem 3.2 $a \in G'$ and so G is totally dot-critical by Lemma 2.2.

$\Rightarrow$ Since G is a totally 2-dot-critical graph, it follows that G is 2-dot-critical. Thus, every component of $\bar{G}$ is spiked or a non-trivial complete graph. Suppose $\bar{G}$ contains a complete component, C, on at least three vertices. Let a, b be two vertices in C. Then in G, $\{a, b\}$ does not dominate the remaining elements of C, and so by Lemma 2.2, one of a or b is critical, say a. But then by Theorem 3.2, a is adjacent to an end vertex in $\bar{G}$, which is impossible. $\square$

For larger values of k, the situation is far more complex and we do not have a characterization for k-dot-critical or k-totally dot-critical graphs for $k\geqslant 3$.

4. Critical vertices

Critical vertices are of great interest in domination theory, and the concept appears in numerous papers on the subject. The set of critical vertices of a dot-critical graph has a number of interesting properties. In fact, it was the study of critical vertices in general that originally motivated the concept of dot-critical. In this section we consider when a dot-critical graph contains critical vertices. Recall that we are always assuming $k\geqslant 2$ throughout the rest of this paper.

It is possible for a 2-dot-critical graph to not have any critical vertices. In fact the following consequence of Theorems 3.2 and 3.4 characterizes the 2-dot-critical graphs with no critical vertices.

Theorem 4.1. *A* 2-*dot-critical graph has no critical vertices if and only if it is complete multipartite with each part containing at least three vertices.*

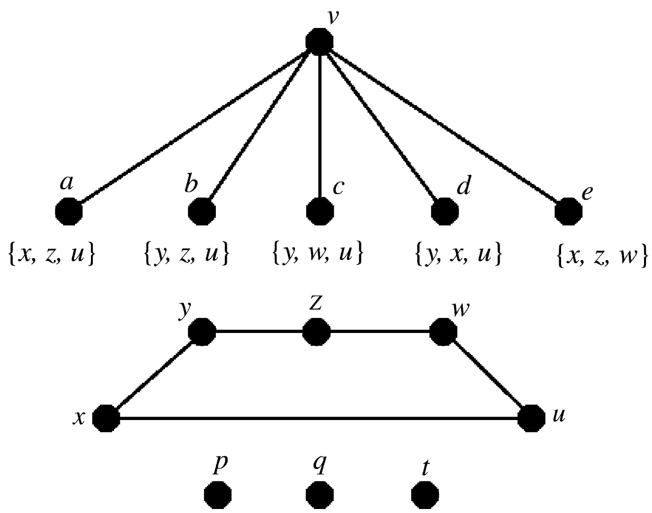

Fig. 1. This graph is 3-dot-critical with diameter three and has no critical verticies. $N(p) = N(q) = N(t) = \{x,y,z,w,u\}$. Removing verticies p, q, and t produces a totally 3-dot- critical graph of diameter no critical verticies.

From Theorems 3.5 and 3.2 it follows that every totally 2-dot-critical graph has $G' \neq \emptyset$. For the case $k \geqslant 3$, things are more difficult.

Lemma 4.2. *If G is a dot-critical graph and $N[v] \subseteq N[u]$, then $v \in G'$.*

Proof. In this case, $G.vu \cong G - v$, and so, $\gamma(G - v) = \gamma(G.vu) = \gamma(G) - 1$. $\square$

Corollary 4.3. *Every end vertex of a dot-critical graph is a critical vertex.*

The next result shows that any connected 3-dot-critical graph with diameter at least four contains a critical vertex. (In general, there are 3-dot-critical graphs with diameter as large as 6 such as the path on 7 vertices.)

Theorem 4.4. *A connected 3-dot-critical graph with $G' = \emptyset$ has a diameter of at most three.*

Proof. Let G be a connected 3-dot-critical graph with no critical vertices and diameter d. Let $v, w \in V(G)$ such that $d(v, w) = d$. Let $A_i = \{x \in V(G) : d(v, x) = i\}$ for $0 \leqslant i \leqslant d$. So $A_0 = \{v\}$. For $x \in A_1$, $v \perp x$ and by Lemma 2.2, there exists an MDS of G containing both v and x, since neither is critical. Thus, $A_3 \cup \cdots \cup A_d$ is dominated by one vertex, z. Hence $d \leqslant 5$. Suppose $d = 5$. Then the vertex z dominates $A_3 \cup A_4 \cup A_5$ which implies that $z \in A_4$. But then for any $x \in A_5$, $N[x] \subseteq N[z]$ and so $x \in G'$ by Lemma 4.2. Thus $d \leqslant 4$. So now suppose that $d = 4$. Let x be an arbitrary vertex of A_1, and let S be an MDS of G that contains both v and x, and let u denote the third element of S. Then clearly u dominates all of $A_3 \cup A_4$. Now if u belongs to A_3, then for any y in A_4, $N[y] \subseteq N[u]$ and so y is critical by Lemma 4.2. Consequently, we have $u \in A_4$. Similarly, if there were some $y \in A_4 - u$, then $N[y] \subseteq N[u]$, and so again, y is critical. Thus, $A_4 = \{u\}$. But the only element of S that can dominate A_2 is x and since x was chosen arbitrarily, that means that every element ofA_1 is adjacent to every element of A_2. By symmetry (since $A_4 = \{u\}$), every element of A_3 is adjacent to every element of A_2. But now, let c be any vertex of A_2 and let d be any vertex of A_3. Then $\{c, d\}$ dominates $G - v$ which is impossible since v is not a critical vertex of G. $\square$

The example in Fig. 1 shows Theorem 4.4 is best possible. In Fig. 1, the diameter of G is three with $A_0 = \{v\}$, $A_1 = \{a, b, c, d, e\}$, $A_2 = \{x, y, z, w, u\}$, and $A_3 = \{p, q, t\}$. Not all edges are shown. For $i \in A_1$, the three vertices in $N(i) \cap A_2$ are written under i.

The next result shows that a totally 3-dot-critical graph with diameter at least three contains a critical vertex.

Theorem 4.5. *A connected totally 3-dot-critical graph with no critical vertices has a diameter of at most two.*

Proof. Let x and y be non-adjacent vertices of G. It is enough to show that x and y have a common neighbor. So, let $A=\{v : v \text{ is not dominated by}\{x, y\}\}$. Then since $\gamma(G)=3$, $A \neq \emptyset$, and so since $G'=\emptyset$, $|A| \geqslant 2$. Let $a, b \in A$. Then there is some vertex z such that $\{a, b, z\}$ dominates G. But then, z dominates both x and y (note that $z \notin \{x, y\}$ since x and y are not adjacent). $\square$

The graph in Fig. 1 minus the vertices p, q, t is totally 3-dot-critical with diameter two and no critical vertices.

Questions.

1. What are the best bounds for the diameter of a k-dot-critical graph and a totally k-dot-critical graph G with $G'=\emptyset$ for $k \geqslant 4$?
2. Is it true that for each $k \geqslant 4$, there exists a k-totally dot-critical graph with no critical vertices?
3. Under what circumstances are k-totally dot-critical graphs critically dominated?

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